# Math 251 Final Exam (Practice)

**Name**: ______________________

**This exam has 12 questions, for a total of 120 points.**

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer. As always, watch out for typos and errors. If you notice any, please let me know.

| Question | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 8 | |
| 3 | 10 | |
| 4 | 8 | |
| 5 | 12 | |
| 6 | 10 | |
| 7 | 12 | |
| 8 | 10 | |
| 9 | 12 | |
| 10 | 8 | |
| 11 | 12 | |
| 12 | 12 | |
| Total: | 120 | |

**Question 1. (6 pts)**

Find all possible values of $a$ so that the plane

$$ax + y = 1$$

forms a angle of 45 degrees with the line

$$\frac{x-1}{2} = \frac{y}{2} = z - 1$$

**Solution:** A normal vector to the plane is

$$\mathbf{n} = \langle a, 1, 0 \rangle$$

and a direction vector of the line is

$$\mathbf{v} = \langle 2, 2, 1 \rangle$$

The angle between the line and the plane is $\pi/4$, therefore the angle between $\mathbf{n}$ and $\pm\mathbf{v}$ is $\pi/2 - \pi/4 = \pi/4$. (**We need either $\mathbf{v}$ or $-\mathbf{v}$ to form an angle of $\pi/4$ with $\mathbf{n}$, since the angle between a plane and a line is always acute.**)

So

$$\frac{\sqrt{2}}{2} = \cos(\pi/4) = \pm\frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}||\mathbf{v}|} = \pm\frac{2a+2}{\sqrt{a^2+1}\sqrt{4+4+1}}$$

Square both sides (and we can get rid of the $\pm$ sign after that) and simplify

$$a^2 - 16a + 1 = 0$$

Therefore, when $a = 8 \pm 3\sqrt{7}$, the plane

$$ax + y = 1$$

forms a angle of 45 degrees with the line

$$\frac{x-1}{2} = \frac{y}{2} = z - 1$$

**Question 2. (8 pts)**

Determine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{4x^2 + 3y^2}$$

exists.

**Solution:** Write

$$f(x, y) = \frac{x^2 + \sin^2 y}{4x^2 + 3y^2}$$

1. Along $x = 0$,

$$f(0, y) = \frac{\sin^2 y}{3y^2}$$

$$\lim_{y\to 0} \frac{\sin^2 y}{3y^2} = \frac{1}{3}$$

The limit of $f(x, y)$ goes to $1/3$ as $(x, y)$ goes to $(0, 0)$ along $x = 0$.

2. Along $y = 0$,

$$f(x, 0) = \frac{x^2}{4x^2} = \frac{1}{4}$$

$$\lim_{y\to 0} f(x, 0) = \frac{1}{4}$$

The limit of $f(x, y)$ goes to $1/4$ as $(x, y)$ goes to $(0, 0)$ along $y = 0$

3. Since $1/3 \neq 1/4$, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{4x^2 + 3y^2}$$

does not exist.

**Question 3. (10 pts)**

Given

$$f(x, y) = x^2 + \sin(xy)$$

(a) Find the directional derivative of $f(x, y)$ in the direction $\langle 1, -1\rangle$ at the point $(1, \pi)$;

(b) Find the tangent plane to the graph of $f(x, y)$ at the point $(1, \pi, 1)$.

**Solution:**

(a)

$$\nabla f = \langle f_x, f_y\rangle = \langle 2x + y\cos(xy), x\cos(xy)\rangle$$
$$\nabla f(1, \pi) = \langle 2 - \pi, -1\rangle$$

Write

$$\mathbf{v} = \langle 1, -1\rangle$$

Then

$$D_{\mathbf{v}} f(1, \pi) = \nabla f(1, \pi) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \langle 2 - \pi, -1\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\rangle = \frac{3 - \pi}{\sqrt{2}}$$

(b) Use the formula

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

From part $(a)$, we know that

$$f_x(1, \pi) = 2 - \pi \quad \text{and} \quad f_y(1, \pi) = -1$$

So an equation of the tangent plane is

$$z - 1 = (2 - \pi)(x - 1) - (y - \pi)$$

**Question 4. (8 pts)**
Use differentials to approximate $\sqrt{0.96} \cdot e^{0.01}$.

**Solution:** Set the function
$$f(x,y) = \sqrt{x}e^y.$$
We shall compare $f(0.96, 0.01) = \sqrt{0.96}e^{0.01}$ with
$$f(1,0) = \sqrt{1}e^0 = 1$$
Compute the differential of $f(x,y)$
$$df = f_x dx + f_y dy = (\frac{1}{2\sqrt{x}}e^y)dx + \sqrt{x}e^y dy$$
At the point $(1,0)$, we have
$$f_x(1,0) = 1/2, f_y(1,0) = 1$$
Moreover, $dx = 0.96 - 1 = -0.04$ and $dy = 0.01 - 0 = 0.01$. So we have
$$df = (1/2) \cdot (-0.04) + 0.01 = -0.01$$
Therefore,
$$\sqrt{0.96} \cdot e^{0.01} \approx f(1,0) + df = 1 - 0.01 = 0.99$$

**Question 5. (12 pts)**

For this question, choose one (and only one) of the following two versions.

**(Version A)** Find the local maximum, minimum and saddle points of

$$f(x, y) = x^3 + y^3 - 3x^2 - 12y$$

**(Version B)** Find the absolute maximum and minimum values of

$$f(x, y) = x^2y + 2x^2 + y^2$$

on $x^2 + 2y^2 = 12$.

**I choose version (circle one) A. B.**

**Solution:**

(**Version** A) First, we need to find all critical points, by solving

$$\begin{cases} f_x = 3x^2 - 6x = 0 \\ f_y = 3y^2 - 12 = 0 \end{cases}$$

So we have solutions $(0, \pm 2)$ and $(2, \pm 2)$.

Now we shall apply the second derivatives test to determine the local max and min's.

$$f_{xx} = 6x - 6, \quad f_{yy} = 6y, \quad f_{xy} = 0$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = (6x - 6)(6y)$$

We have

(1) $D(0, 2) = -72 < 0$, so $(0, 2)$ is a saddle point

(2) $D(0, -2) = 72 > 0$, and $f_{xx}(0, -2) = -6 < 0$, so $(0, -2)$ is a local max

(3) $D(2, 2) = 72 > 0$, and $f_{xx}(2, 2) = 6 > 0$, so $(2, 2)$ is a local min

(4) $D(2, -2) = -72 < 0$, so $(2, -2)$ is a saddle point

(**Version** B) We shall apply Lagrange multiplier method. write

$$g(x, y) = x^2 + 2y^2 = 12$$

(1) Solve

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + 2y^2 = 12 \end{cases}$$

i.e.

$$\begin{cases} 2xy + 4x = \lambda(2x) \Longrightarrow 2x(y + 2) = \lambda(2x) \\ x^2 + 2y = \lambda(4y) \\ x^2 + 2y^2 = 12 \end{cases}$$

From the first equation, we see that either $x = 0$ or $y + 2 = \lambda$.

(i) If $x = 0$, then the last equation implies that $y = \pm\sqrt{6}$. With a little checking of the second equation, we have the two solutions $(0, \pm\sqrt{6})$

(ii) If $x \neq 0$, then $y + 2 = \lambda$. Substitute this into the second equation, we have

$$x^2 + 2y = (y + 2)(4y) \Longrightarrow x^2 = 4y^2 + 6y$$

plug this into the third equation, we have

$$4y^2 + 6y + 2y^2 = 12$$

so $y = -2$ (which implies $x = \pm 2$) or 1 (which implies $x = \pm\sqrt{10}$). we have another four solutions

$$(\pm\sqrt{10}, 1) \quad \text{and}(\pm 2, -2)$$

(iii) compare

$$f(0, \pm\sqrt{6}) = 6, f(\pm\sqrt{10}, 1) = 31 \quad \text{and} \quad f(\pm 2, -2) = 4$$

On the ellipse $x^2 + 2y^2 = 12$, the absolute max of $f$ is $f(\pm\sqrt{10}, 1) = 31$ and the absolute min of $f$ is $f(\pm 2, -2) = 4$.

**Question 6. (10 pts)**

Given the triple integral

$$\iiint_E (x^2 + z^2)\ dV$$

where $E$ is the part of the unit ball in the first octant

(a) write the integral in $xyz$ coordinates.

(b) write the integral in spherical coordinates.

**Solution:**

(a)

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + z^2)\ dzdxdy$$

(b)

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \cos^2 \varphi)\rho^2 \sin \varphi\ d\rho d\varphi d\theta$$

**Question 7. (12 pts)**

(a) Determine if

$$\mathbf{F}(x,y,z) = \langle 2x + e^x z,\ \sin y,\ e^x \rangle$$

is a conservative vector field. If it is, find a function $f$ such that $\nabla f = \mathbf{F}$.

(b) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $C$ is the curve

$$\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 1$$

**Solution:**

(a) The domain of $\mathbf{F}$ is $\mathbb{R}^3$, which is simply-connected.

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + e^x z & \sin y & e^x \end{vmatrix} = \langle 0, -(e^x - e^x), 0 \rangle = \mathbf{0}$$

So $\mathbf{F}$ is conservative.

To find $f$ such that $\nabla f = \mathbf{F}$. First,

$$f_x = 2x + e^x z \Longrightarrow f(x,y,z) = x^2 + e^x z + g(y,z)$$

Then

$$f_y = 0 + 0 + g_y = \sin y \Longrightarrow g(y,z) = -\cos y + h(z)$$

which implies that

$$f(x,y,z) = x^2 + e^x z - \cos y + h(z)$$

Now

$$f_z = 0 + e^x - 0 + h'(z) = e^x \Longrightarrow h(z) = C.$$

Therefore

$$f(x,y,z) = x^2 + e^x z - \cos y + C$$

for any constant $C$.

(b) By the fundamental theorem of line integral, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = (1/e + C) - C = 1/e$$

(In fact, you can pick a number for $C$, say $C = 1$ in the expression

$$f(x,y,z) = x^2 + e^x z - \cos y + C$$

In any case, $C$ will be cancelled out in the end. ) Notice that $\mathbf{r}(0) = (1,0,0)$ and $\mathbf{r}(1) = (-1,0,1)$.

**Question 8. (10 pts)**

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \langle y,\ x,\ z\rangle$ and $S$ is the part of the paraboloid $z = 2 - x^2 - y^2$ above the plane $z = 1$. Assume $S$ is oriented downward.

**Solution:** Warning: $S$ only consists of the part from the paraboloid. The bottom disk is not included. So we cannot apply the divergence theorem here.

Parametrize $S$ by

$$\mathbf{r}(x, y) = \langle x, y, 2 - x^2 - y^2\rangle$$

with domain $D$ enclosed by the curve $1 = 2 - x^2 - y^2$, that is, $D$ is $x^2 + y^2 \leq 1$. Then

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle$$

which gives the opposite of the given (downward) orientation.

$$\begin{aligned}
&\iint_S \mathbf{F} \cdot d\mathbf{S} \\
&= -\iint_D \langle y, x, 2 - x^2 - y^2\rangle \cdot \langle 2x, 2y, 1\rangle dA \\
&= -\iint_D 4xy + 2 - (x^2 + y^2)\ dA \\
&= -\int_0^{2\pi} \int_0^1 (4r^2 \sin\theta\cos\theta + 2 - r^2) r\,dr\,d\theta \\
&= -\int_0^{2\pi} (\sin\theta\cos\theta + \frac{3}{4}) d\theta = -\frac{3\pi}{2}
\end{aligned}$$

**Question 9. (12 pts)**

Evaluate

$$\oint_C xy\ dx$$

where $C$ is the closed curve that consists of the upper half of the unit circle $x^2 + y^2 = 1$ and the part of the parabola $y = x^2 - 1$ below the $x$-axis. Assume $C$ is oriented counterclockwise.

**Solution:** Use Green's theorem (denote by $D$ the region enclosed by $C$, then $C$ is positively oriented as the boundary of $D$), then we take $P = xy$ and $Q = 0$,

$$\begin{aligned}
\oint_C xy\ dx &= \iint_D -x dA \\
&= \int_{-1}^{1}\int_{x^2-1}^{\sqrt{1-x^2}} -x dy dx \\
&= -\int_{-1}^{1} x(x^2-1) - x\sqrt{1-x^2} dx \\
&= -\int_{-1}^{1} x(x^2-1) dx + \int_{-1}^{1} x\sqrt{1-x^2} dx \\
&= 0 + (-1/2)\int_0^0 u^{1/2} du \quad (\text{where } u = 1 - x^2) \\
&= 0
\end{aligned}$$

**Question 10. (8 pts)**

Evaluate

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \cos(x^2+y^2)dydx + \int_1^2 \int_0^{\sqrt{4-x^2}} \cos(x^2+y^2)dydx.$$

**Hint: you need to rewrite the integral.**

**Solution:** First we need to figure out the domain of integration. After putting the two pieces together (you should draw a picture), we see that the domain is an annulus in the first quadrant. So in polar coordinates, the two integrals can be combines together into one as follows:

$$\begin{aligned}\int_0^{\pi/2} \int_1^2 \cos(r^2) r dr d\theta &= \int_0^{\pi/2} \left.\frac{\sin(r^2)}{2}\right|_1^2 d\theta \\ &= \frac{\sin(4)-\sin(1)}{4} \cdot \pi\end{aligned}$$

**Question 11. (12 pts)**

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = \langle 2e^x y, -e^x y^2 + y, z + \cos x \rangle$ and $S$ is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $x + y + z = 1$. Assume $S$ is oriented outward.

**Solution:** Use divergence theorem (note $S$ is positively oriented)

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div}\mathbf{F} \; dV \\ &= \iiint_D (2e^x y - 2e^x y + 1 + 1) \; dV = \iiint_E 2 \; dV \end{aligned}$$

use cylindrical coordinates

$$\begin{aligned} &= \int_0^{2\pi} \int_0^1 \int_{-1}^{1 - r\cos\theta - r\sin\theta} 2r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 2(2 - r\cos\theta - r\sin\theta) r dr d\theta = \cdots = 4\pi \end{aligned}$$

**Question 12. (12 pts)**

Evaluate

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = e^{xy}\sin z\mathbf{i} + xz^2\mathbf{j} + yz\mathbf{k}$ and $S$ is the hemisphere $x = \sqrt{1 - y^2 - z^2}$, oriented towards the positive $x$-axis.

**Solution:** Use Stoke's theorem, the boundary of $S$ is the unit circle $y^2 + z^2 = 1$ in $yz$-plane, oriented counterclockwise. Parametrize $C$ by

$$\mathbf{r}(\theta) = \langle 0, \cos\theta, \sin\theta\rangle$$

with $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \langle \sin(\sin\theta), 0, \cos\theta\sin\theta\rangle \cdot \langle 0, -\sin\theta, \cos\theta\rangle \; d\theta \\ &= \int_0^{2\pi} \cos^2\theta\sin\theta d\theta = \cdots = 0\end{aligned}$$

**Alternatively**, Use Stoke's theorem again, but this time choose $S_1$ to be the disk $y^2 + z^2 \leq 1$ is $yz$-plane, oriented towards the positive $x$-axis. Then

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S}$$

Now

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy}\sin z & xz^2 & yz \end{vmatrix} = \langle z - 2xz, e^{xy}\cos z, z^2 - xe^{xy}z^2\rangle$$

The surface $S_1$ can be parametrized by

$$\mathbf{r}(y, z) = \langle 0, y, z\rangle$$

with domain $D : y^2 + z^2 \leq 1$.

$$\mathbf{r}_y \times \mathbf{r}_z = \langle 1, 0, 0\rangle \text{ agrees with the chosen orientation}$$

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D z dA = \int_0^{2\pi}\int_0^1 r\sin\theta \; r dr d\theta = \cdots = 0$$